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Short communication

A nonlinear model to generate the winner-take-all competition

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ABSTRACT

This paper is concerned with the phenomenon of winner-take-all competition. In this paper, we propose a continuous-time dynamic model, which is described by an ordinary differential equation and is able to produce the winner-take-all competition by taking advantage of selective positive–negative feedback. The global convergence is proven analytically and the convergence rate is also discussed. Simulations are conducted in the static competition and the dynamic competition scenarios. Both theoretical and numerical results validate the effectiveness of the dynamic equation in describing the nonlinear phenomena of winner-take-all competition.

1. Introduction

Competition widely exists in nature and the society. Among different kinds of competitions, winner-take-all competition refers to the phenomena that individuals in a group compete with each others for activation and only the one with the highest input stays activated while all the others deactivated. Examples of this type of competition include the dominant growth of the central stem over others [1], the contrast gain in the visual systems through a winner-take-all competition among neurons [2], competitive decision making in the cortex [3,4], cell fate competition [5,6], etc.

Although many phenomena, as exemplified above, demonstrate the same winner-take-all competition, they may have different underlying principles in charge of the dynamic evolution. There are various mathematic models presented to describe this type of competition phenomena, e.g., the N species Lotka–Volterra model [7,8], interactively spiking FitzHugh–Nagumo Model [9–11], optimization based model [12,13], discrete-time different equation model [14], neural network model [15,16], lateral inhabitation model [17,18], to name a few. However, these models are often very complicated due to the compromise with experimental realities in the particular fields. Consequently, the essence of the winner-take-all competition may be embedded in the interaction dynamics of those models, but difficult to tell from the sophisticated dynamic equations. Motivated by this, a simple ordinary differential equation model with a direct and intuitive explanation is presented in this paper and it is expected to cast lights to researchers on the principle of competition phenomena in different fields.

The remainder of this paper is organized as follows: in Section 2, the analytical model is presented and the underlying competition mechanism is explained from a selective positive–negative feedback perspective. In Section 3, the competition

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behavior and the convergence results are proven rigourously by means of nonlinear stability tools. In Section 4, illustrative examples are given to show the effectiveness of the proposed model. The paper is concluded in Section 5.

2. The model

The proposed model has the following dynamic for the ith agent in a group of totally n agents,

$$\dot{x}_i = c_0(u_i - \|x\|^2)x_i \tag{1}$$

where $x_i \in \mathbb{R}$ denotes the state of the i agent, $u_i \in \mathbb{R}$ is the input and $u_i \geqslant 0$, $u_i \neq u_j$ for $i \neq j$, $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ denotes the Euclidean norm of the state vector $x = [x_1, x_2, \ldots, x_n]^T$, $c_0 \in \mathbb{R}$ $c_0 \geqslant 0$ is a scaling factor.

The dynamic Eq. (1) can be written into the following compact form by stacking up the state for all agents,

$$\dot{x} = c_0(u \circ x - \|x\|^2 x) \tag{2}$$

where $x = [x_1, x_2, \ldots, x_n]^T$, $u = [u_1, u_2, \ldots, u_n]^T$, the operator '$\circ$' represents the multiplication in component-wise, i.e., $u \circ x = [u_1x_1, u_2x_1, \ldots, u_nx_n]^T$.

Remark 1. In the dynamic Eq. (1), all quantities on the right hand side can be obtained locally from the ith agent itself (u_i and x_i) except the quantity $\|x\|^2$, which reflects the effort from other agents over the ith one (as sketched in Fig. 1). Actually, $\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ is the second moment about the origin of the group of agents and it is a statistic of the whole group. In this regard, the dynamic model (1) implies that the winner-take-all competition between agents may emerge in a multi-agent system if each agent accesses the global statistic $\|x\|^2$ (instead of exactly knowing states of all the other agents) besides its own information.

As will be stringently demonstrated later, the agent with the largest input will finally win the competition and keep active while all the other agents will be deactivated to zero eventually. Before proving this result rigorously, we present a intuitive explanation of the result in a sense of positive feedback vs. negative feedback. Note that the term $c_0u_ix_i$ in Eq. (1) provides a positive feedback to the state variable x_i as $u_i \geqslant 0$ while the term $-c_0\|x\|^2x_i$ supplies a negative feedback. For the ith agent, if $u_i = \|x\|^2, x_i$ will keep the value. If $u_i < \|x\|^2$, the positive feedback is less than the negative feedback in value and the state value attenuates to zero. In contrast, if $u_i > \|x\|^2$, the positive feedback is greater than the negative feedback and the state value tends to increase as large as possible until the resulting increase of $\|x_i\|$ surpasses u_i. Particularly for the winner, say the k^*th agent, $u_{k^*} > u_i$ holds for all $i \neq k^*$. In this case, all agents have negative feedbacks and keep reducing in values until $\|x\|^2$ reduces to the value of u_k when $u_k < \|x\|^2$. Otherwise when u_k is slightly greater than $\|x\|^2$ (by slightly greater we mean $u_k > \|x\|^2 > u_l$ with l denoting the agent with the second largest state value), only the winner has a positive feedback and has an increase in its state value while all the other agents have negative feedbacks and keep reducing until $\|x\|^2$ equals u_k. Under this selective positive–negative feedback mechanism, the winner finally stays active at the value $u_{k^*} = \|x\|^2$ while the losers are deactivated to zero.

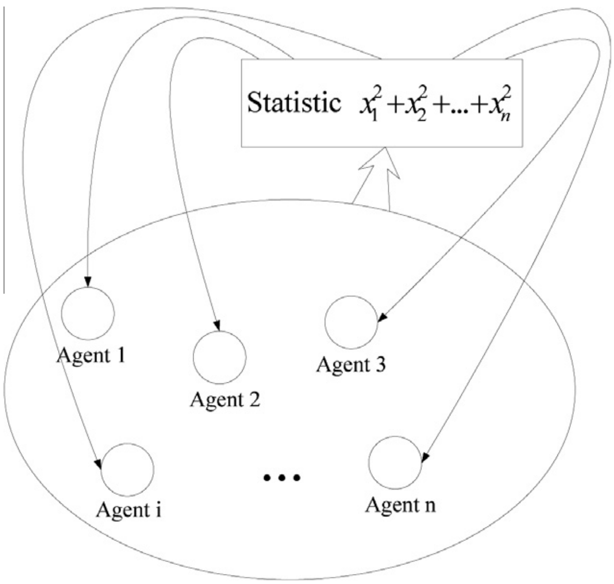

Fig. 1. Information flow for the agent dynamics.

3. Theoretical analysis and results

In this section, theoretical results on the dynamic system (1) are presented. The rigorous proof of the main results needs the uses of LaSalle's invariant set principle [19,20], local stability analysis and the ultimate boundedness theory [21].

Lemma 1 [21]. *Let $\mathbb{D} \subset \mathbb{R}^n$ be a domain that contains the origin and $V : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ be a continuous differentiable function such that*

$$\alpha_1(\|x\|) \leqslant V(t,x) \leqslant \alpha_2(\|x\|) \tag{3}$$

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leqslant -W(x), \quad \forall \|x\| \geqslant \mu > 0 \tag{4}$$

$\forall t \geqslant 0$ and $\forall x \in \mathbb{D}$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W(x)$ is a continuous positive definite function. Take $r > 0$ such that $\mathbb{B}_r \subset \mathbb{D}$ and suppose that $\mu < \alpha_2^{-1}(\alpha_1(r))$. Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leqslant \alpha_2^{-1}(\alpha_1(r))$, there is $T \geqslant 0$ (dependent on $x(t_0)$ and μ) such that the solution of $\dot{x} = f(t,x)$ satisfies,

$$\|x(t)\| \leqslant \alpha_1^{-1}(\alpha_2(\mu)) \quad \forall t \geqslant t_0 + T \tag{5}$$

Moreover, if $\mathbb{D} = \mathbb{R}^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then the result (6) holds for any initial state $x(t_0)$, with no restriction on how large μ is.

With Lemma 1, we are able to prove the following lemma for our main result,

Lemma 2. *There exists $T \geqslant 0$ (dependent on $x(t_0)$ and μ) such that the solution of the agent dynamic Eq. (2) satisfies,*

$$\|x(t)\| \leqslant \mu \quad \forall t \geqslant t_0 + T \tag{6}$$

where $\mu = \sqrt{\max\{u_1, u_2, \ldots, u_n\}} + \delta$ with $\delta > 0$ being any positive constant.

Proof. We prove the result by following the framework of Lemma 1. Let $\mathbb{D} = \mathbb{R}^n$, $V = \frac{1}{2}x^T x$ and $\alpha_1(\|x\|) = \alpha_2(\|x\|) = \frac{1}{2}\|x\|^2 = V$. For V, we have,

$$\dot{V} = x^T \dot{x} = c_0 x^T (u \circ x - \|x\|^2 x) = c_0 x^T \left(diag(u)x - \|x\|^2 x\right) = c_0 x^T \left(diag(u) - \|x\|^2\right) x \leqslant c_0(u_0 - \|x\|^2) x^T x \tag{7}$$

The equation $u \circ x = \mathrm{diag}(u)x$ is used in the second step of the above derivation. Note that $\left(diag(u) - \|x\|^2\right)$ is a diagonal matrix and its largest eigenvalue is $u_0 - \|x\|^2$. Therefore, $x^T\left(diag(u) - \|x\|^2\right)x \leqslant (u_0 - \|x\|^2)x^T x$, from which the last inequality in (7) is obtained. As $u_i \geqslant 0$ for all i and $u_i \neq u_j$ for $i \neq j$, we get $u_0 > 0$. Recall $\mu = \sqrt{\max\{u_1, u_2, \ldots, u_n\}} + \delta$, i.e., $\mu = \sqrt{u_0} + \delta$ for any small positive $\delta > 0$. For $\|x\| \geqslant \mu$, $u_0 - \|x\|^2 \leqslant -\delta^2$. Together with (7), we get,

$$\dot{V} \leqslant -c_0 \delta^2 x^T x \tag{8}$$

for $\|x\| \geqslant \mu$. Choosing a positive definite function $W(x) = c_0\delta^2 x^T x$ yields $\dot{V} \leqslant -W(x)$ for $\forall \|x\| \geqslant \mu$. Therefore, according to Lemma 1, there exists $T \geqslant 0$ such that the solution satisfies $\|x(t)\| \leqslant \alpha_1^{-1}(\alpha_2(\mu)) = \mu$, $\forall t \geqslant t_0 + T$. This completes the proof. □

Remark 2. Lemma 2 means the state of the dynamic model (2) is ultimately bounded inside a compact super ball in $\mathbb{R}^n$ with radius $\mu = \sqrt{\max\{u_1, u_2, \ldots, u_n\}} + \delta$. In other words, this super ball is positively invariant with respect the system dynamic (2). This result allows us to apply LaSalle's invariant set principle for further investigation of the system behaviors.

Lemma 3 [19]. *Let $\Omega \subset \mathbb{D}$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V : \mathbb{D} \to \mathbb{R}$ be a C^1-function such that $\dot{V}(x) \leqslant 0$ on Ω. Let $\mathbb{E}$ be the set of all points in Ω such that $\dot{V}(x) = 0$. Let $\mathbb{M}$ be the largest invariant set in $\mathbb{E}$. Then, every solution starting in Ω approaches $\mathbb{M}$ as $t \to \infty$.*

Remark 3. It is worth noting that the mapping V in Lemma 3 is not necessary to be positive definite, which is a major difference from the Lyapunov function in conventional stability analysis of dynamic systems [19]. Instead, V is required to be be a continuous differentiable function in Lemma 3, which is much looser than the positive definite requirement and simplifies the analysis.

Theorem 1. *The solution of the system involving n dynamic agents with the ith agent described by (1) globally approaches 0 for $i \neq k^*$ and approaches $\sqrt{u_{k^*}}$ or $-\sqrt{u_{k^*}}$ for $i = k^*$ as $t \to \infty$, where k^* denotes the label of the winner, i.e., $k^* = \mathrm{argmax}\{u_1, u_2, \ldots, u_n\}$.*

Proof. There are two steps for the proof. The first step is to prove that the state variable ultimately converges to a set consisting of a limit number of points and the second step proves there is only a single point among the candidates is stable.

Step 1: According to Lemma 2, the state variable x in the system dynamic (2) is ultimately bounded by a compact super ball in $\mathbb{R}^n$ with radius $\mu = \sqrt{\max\{u_1, u_2, \ldots, u_n\}} + \delta$, which implies this super ball is positively invariant with respect the system dynamic (2) and the super ball $\{x \in \mathbb{R}^n | \|x\| \leqslant \mu\}$ is qualified to be the set Ω in Lemma 3.

Let $V = -\frac{1}{2}x^T \mathrm{diag}(u)x + \frac{1}{4}\|x\|^4$. Apparently, V is a C^1-function. For V, we have,

$$\dot{V} = -x^T \mathrm{diag}(u)\dot{x} + \|x\|^2 x^T \dot{x} = \left(-x^T \mathrm{diag}(u) + \|x\|^2 x^T\right)\dot{x} \tag{9}$$

With $x^T \mathrm{diag}(u) = (x \circ u)^T$, we get $x^T \mathrm{diag}(u) - \|x\|^2 x^T = (x \circ u - \|x\|^2 x)^T$. Together with (9), we have,

$$\dot{V} = -c_0 (x \circ u - \|x\|^2 x)^T (x \circ u - \|x\|^2 x) = -c_0 \|x \circ u - \|x\|^2 x\|^2 \leqslant 0 \tag{10}$$

We find $\mathrm{diag}(u)x = \|x\|^2 x$ by letting $\dot{V} = 0$. Note that $\mathrm{diag}(u)x = \|x\|^2 x$ is actually a eigenvector equation relative to the matrix $\mathrm{diag}(u)$. The solution can be solved as the set $\mathbb{M} = \{0, \pm\sqrt{u_i}e_i \text{ for } i = 1, 2, \ldots, n\}$, where e_i is a n-dimensional vector with the ith component 1 and all the other component 0. According to Lemma 3, every solution starting in $\Omega = \{x \in \mathbb{R}^n | \|x\| \leqslant \mu\}$ approaches $\mathbb{M}$ as $t \to \infty$. Together with the fact proven in Lemma 2 that every solution stays in Ω ultimately, we conclude that every solution with the initialization $x(t_0) \in \mathbb{R}^n$ approaches $\mathbb{M}$ as $t \to \infty$.

Step 2: We have shown that there are several candidate fixed points to stay for the dynamic system. In this step, we show that all those fixed points in $\mathbb{M}$ are unstable except $x = \pm\sqrt{u_k}e_k$, where $k^* = \mathrm{argmax}\{u_1, u_2, \ldots, u_n\}$. Lyapunov's indirect method suffices the analysis of the un-stability.

For the fixed point $x_e = 0$, the system dynamic (2) is linearized as $\dot{x} = c_0 \mathrm{diag}(u)x$ about $x = 0$ and is unstable as the eigenvalues of the system matrix $c_0 \mathrm{diag}(u)$ have positive real parts.

For the fixed points $x_e = \pm\sqrt{u_i}e_i$, the linearized system around the fixed point is as follows,

$$\dot{x} = c_0 \left(\mathrm{diag}(u) - 2x_e x_e^T - \|x_e\|^2\right) x \tag{11}$$

The system matrix of the above system is a diagonal matrix and its jth diagonal component, which is also its jth eigenvalue, is $c_0(u_j - u_i)$ for $j \neq i$ and $-2c_0$ for $j = i$. Clearly, all the eigenvalues have negative real part only when $u_j - u_i < 0$ holds for all $j \neq i$, i.e., when $i = k^*$, which excludes all fixed points except for $x_e = \pm\sqrt{u_{k^*}}e_{k^*}$ from the stable ones.

In summary, we conclude that every solution approaches $x = \pm\sqrt{u_{k^*}}e_{k^*}$ ultimately with $k^* = \mathrm{argmax}\{u_1, u_2, \ldots, u_n\}$ and e_{k^*} being a n-dimensional vector with the k^*th component 1 and all the other component 0. Entrywisely, the solution approaches $x_i = 0$ for $i \neq k^*$ and $x_{k^*} = \pm\sqrt{u_{k^*}}$, which completes the proof. □

Remark 4. According to Theorem 1, the steady-state value of the winner is either $\sqrt{u_{k^*}}$ or $-\sqrt{u_{k^*}}$. Actually, we can conclude that it is $\sqrt{u_{k^*}}$ if the initial state of the winner is positive while it is $-\sqrt{u_{k^*}}$ if the initial value is negative by noting that $\dot{x}_{k^*} = 0$ when $x_{k^*} = 0$ in (1) for $i = k^*$, which means the state value x_{k^*} will never cross the critical value $x^* = 0$.

4. Illustrative examples

In this section, simulations are provided to illustrate the the winner-take-all competition phenomena generated by the agent dynamic (1). We consider two sceneries: one is static competition, i.e., the input u is constant and one is dynamic competition, i.e., the input u is time-varying.

4.1. Static competition

For the static competition problem, we consider time invariant signals as the input. In the simulation, we consider a problem with $n = 15$ agents. The input u is randomly generated between 0 and 1, which is $u = [0.0924, 0.0078, 0.4231, 0.6556, 0.7229, 0.5312, 0.1088, 0.6318, 0.1265, 0.1343, 0.0986, 0.1420, 0.1683, 0.1962, 0.3175]$, and the state is randomly initialized between -1 and 1, which is $x(0) = [0.7556, 0.1649, -0.8586, 0.8455, 0.6007, -0.4281, 0.0873, 0.9696, 0.4314, 0.6779, -0.1335, -0.0588, 0.1214, -0.4618, 0.4980]$. In the simulation, we choose the scaling factor $c_0 = 1$. Fig. 2 shows the evolution of state values of all agents with time, from which it can be observed that only a single state (corresponds to the 5th agent, which has the largest value in u) has a non-zero value eventually and all the other state values are suppressed to zero. Also, the value of x_5 approaches $\sqrt{u_5}$ (see Fig. 2), which is consistent with the claim made in Remark 4 since the initial value $x_5(0) = 0.6007 > 0$.

To fully visualize the interaction between agents, we consider a three agent case with $u = [0.7368, 0.2530, 0.4117]$. Fig. 3 shows the phase plot of the state in three-dimensional space and its projections in two-dimensional space. Clearly, we can see that the states with the initial state value of the winner being negative (i.e., $x_1(0) < 0$) is attracted to $[-\sqrt{u_1}, 0, 0]$ while is attracted to $[\sqrt{u_1}, 0, 0]$ for the cases with positive initial state values of the winner (i.e., $x_1(0) > 0$). It is worth noting that $x_1(t)$ appears staying at 0 in the situation with $x_1(0) = 0$ in Fig. 3, which seems in contradiction with the statement that the winner $x_1(t)$ converges to either $\sqrt{u_1}$ or $-\sqrt{u_1}$ eventually. Actually, as mentioned in the proof of the Theorem 1, all fixed points are

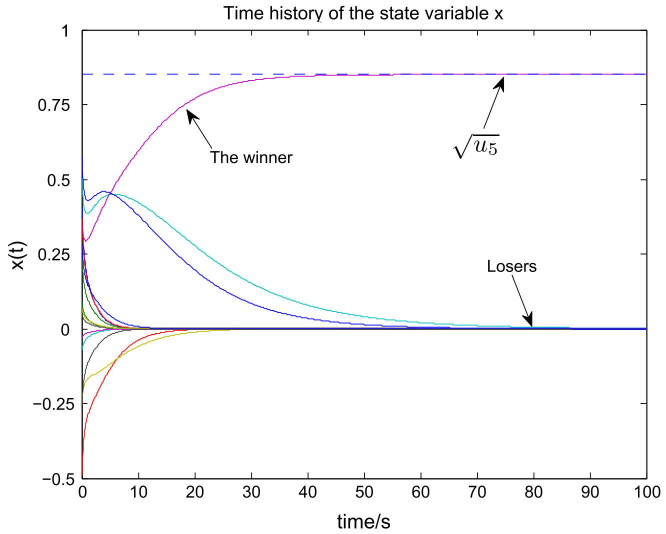

Fig. 2. Agent state trajectories in the static competition scenario with 15 agents.

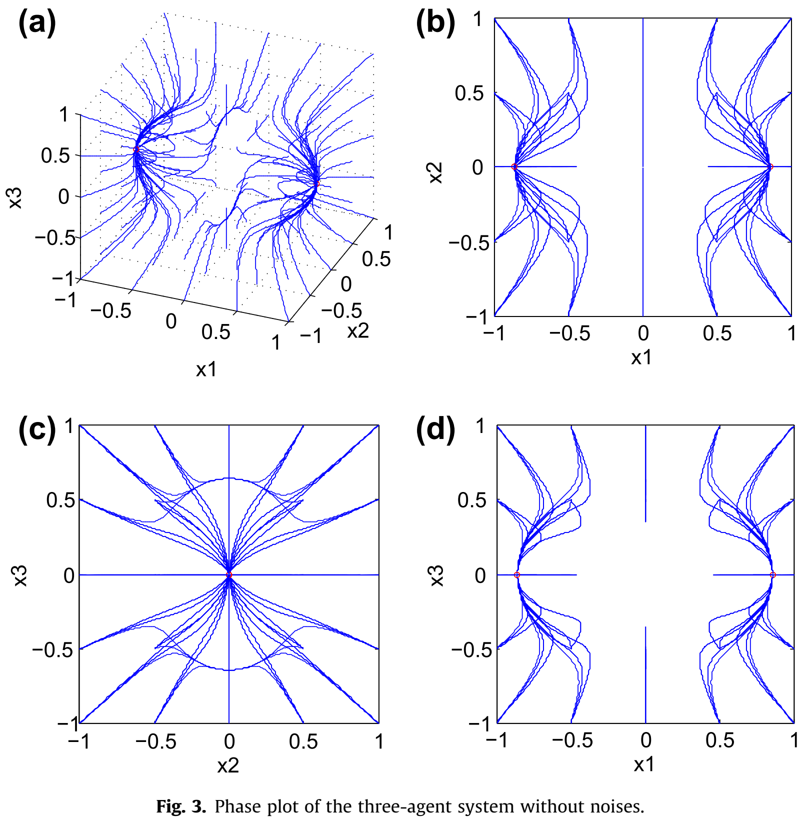

Fig. 3. Phase plot of the three-agent system without noises.

unstable except $\pm\sqrt{u_{k^*}}e_{k^*}$ (k^* denotes the label of the winner and e_{k^*} is a n dimensional vector with the k^*th element being 1 and all the other elements being zeros). Therefore, in this case, the state with $x_1 = 0$ is unstable and must be very subjective to disturbances. To show this, we plug a small random Gaussian white noise with zero mean and 0.0001 variance into the agent dynamic (1). In equation, the resulting dynamic for the ith agent is $\dot{x}_i = c_0(u_i - \|x\|^2)x_i + 0.0001\,v_i$, where v_i is a Gaussian white noise with zero mean unit variance and it is independent with v_j for $j \neq i$. Even with such a small perturbation with a magnitude of 0.0001, the states with $x_1(0) = 0$ either converge to $[\sqrt{u_1}, 0, 0]$ or $[-\sqrt{u_1}, 0, 0]$ instead of staying at $x_1 = 0$ as shown in Fig. 4 and Fig. 5, where the state is initialized at $x(t_0) = 0 \times [-1, -0.5, 0, 0.5, 1]^2$.

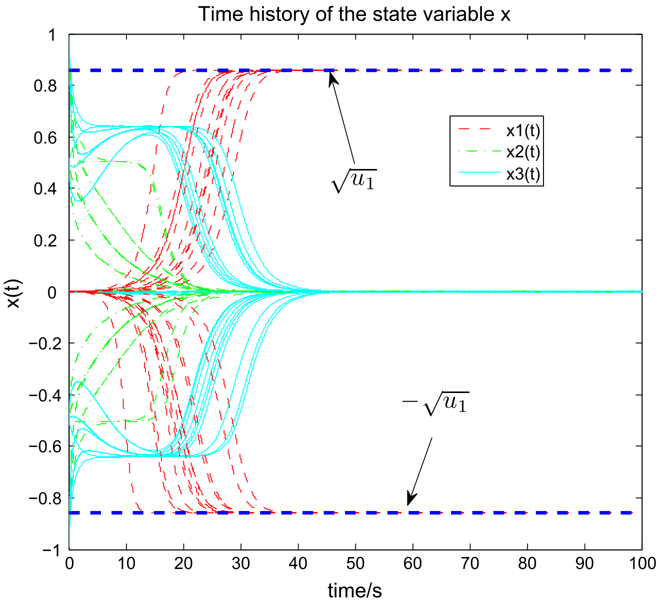

Fig. 4. Time history of the three-agent system with small perturbations.

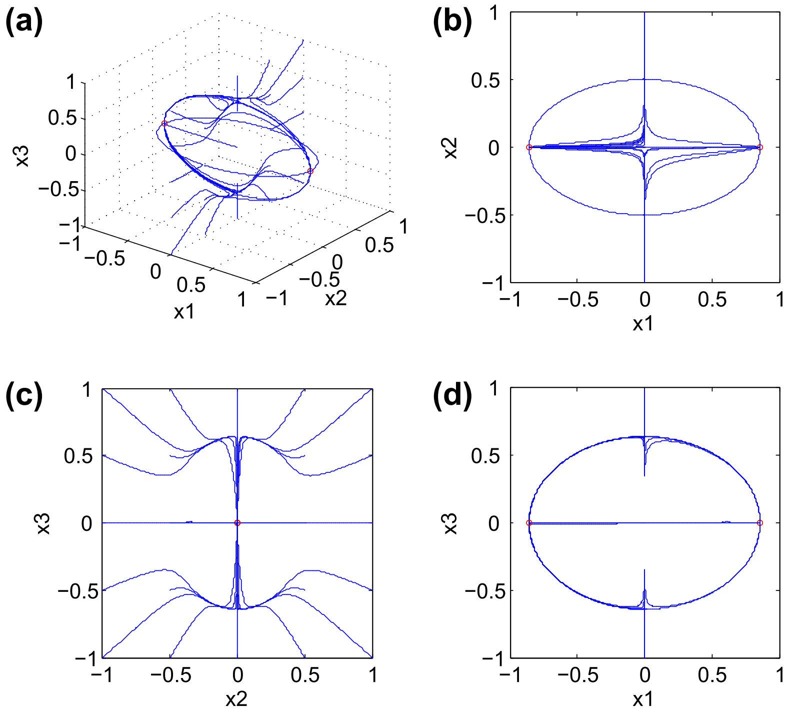

Fig. 5. Phase plot of the three-agent system with small perturbations.

4.2. Dynamic competition

In this part, we consider the scenario with time-varying inputs. For the dynamic system (1), the convergence can be accelerated by choosing a large scaling factor c_0, and the resulting fast response allows the computation of x in real time with time-varying input $u(t)$. In this simulation, we choose $c_0 = 10^4$ and consider $n = 4$ agents with input $u_i(t) = 1 + \sin(2\pi t + 0.25i)$ for $i = 1, 2, 3, 4$, respectively. The initial state valued are randomly generated between -1 and 1. The four

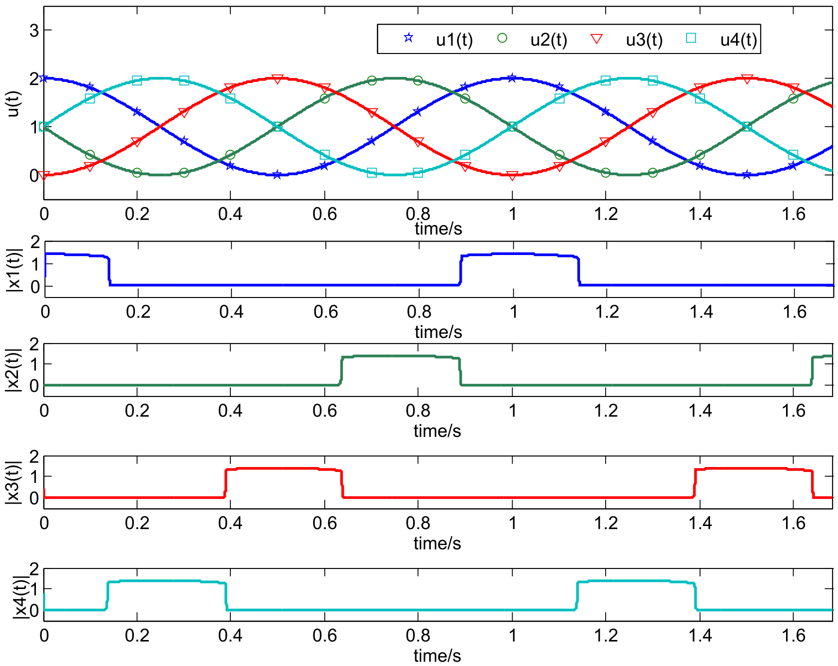

Fig. 6. Inputs and outputs of the dynamic system in the dynamic competition scenario.

input signals and the absolute value of the state variables are plotted in Fig. 6. From this figure, we can see the system can successfully find the winner in real time.

5. Conclusions

In this paper, the winner-take-all competition among agents in a group is considered and an ordinary differential equation describing the dynamics of each agent is proposed. In contrast to existing models, this dynamic equation features a simple expression and an explicit explanation of the competition mechanism, which is expected to help researchers gain some insights into the winner-take-all phenomena in their specialized fields. The fact that the state value of the winner converges to be active while the others deactivated is proven theoretically. The convergence rate is discussed based on a local approximation. Simulations with both static inputs and dynamic inputs are performed. The results validate the effectiveness of the dynamic equation in describing the nonlinear phenomena of winner-take-all competition.

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